Star Formation 2020

Q&A 12.05.2020

Worked Example - Cloud Collapse and Fragmentation:

a) Jeans scale - repetition from the notes

Estimate the length scale at which the gravitational forces balance the thermal pressure, for an isolated cloud of gas with temperature T, density ρ and molecular weight μ (this is a crude estimate of the Jeans length). Convert this to a mass scale, which is called the Jeans mass.

There are many ways to derive this length scale, one of them is to use the Virial theorem to calculate the size of a cloud that can balance its self-gravitational energy with its thermal energy:

$$2\,E_{thermal} + E_{grav} = 0 \quad \Rightarrow \quad 2 \times \frac{3}{2}\,N\,k_B\,T = \frac{3}{5}\,\frac{G\,M^2}{R}$$

here N is the total number of particles in the cloud $N = \frac{M}{\mu m_H} = \frac{4}{3}\pi R^3 \frac{\rho}{\mu m_H}$

It follows:

$$R_J = \left(\frac{15}{4\pi}\,\frac{k_B\,T}{G\,\rho\,\mu\,m_H}\right)^{1/2}$$

$$M_J = \left(\frac{5\,k_B\,T}{G\,\mu\,m_H}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}$$

b) Effects beyond the balance

What happens to a gas cloud smaller than this length scale? how about larger?

Assuming that all the other parameters remain unaffected, for a cloud smaller than R_J the gravitational and thermal energy follow that $|E_{grav}| > 2\,E_{thermal}$, and hence the cloud will be unbound.

On the other hand, if $|E_{grav}| < 2\,E_{thermal}$ (and hence $R > R_J$) the cloud is bound. Since the gravitational energy scales like R^5, while the thermal energy scales like R^3, the gravitational force will increase more rapidly than the thermal force, and we can have collapse.

c) Minimum collapse mass

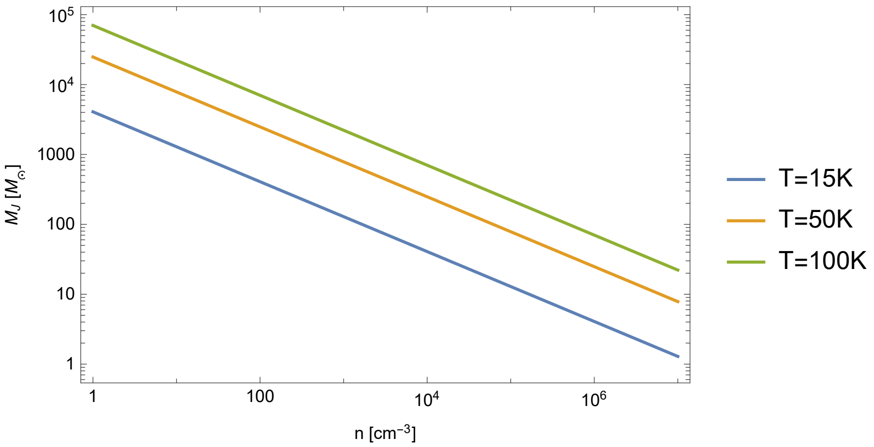
Giant Molecular Clouds (GMC) have typical temperatures of $T = 15$ K, densities of $n = 100\ \mathrm{cm}^{-3}$ and masses ranging from $10^5 - 10^6\,M_\odot$. For this temperature and density, what is the mass of the smallest gas cloud for which gravity overwhelms internal pressure?

In[8]:= $\left(\frac{5\times 1.38\times 10^{-16}\times 15}{6.67\times 10^{-8}\times 1.2\times 1.67\times 10^{-24}}\right)^{3/2}\left(\frac{3}{4\,\pi\,100\times 1.67\times 10^{-24}}\right)^{1/2} * \frac{1}{2\times 10^{33}}$

Out[8]= 407.326

For $T = 15\,K$ and $n = 100\ \text{cm}^{-3}$, the Jean mass is: $M_J = 400\,M_\odot$ (this result might vary according to the approximations you used when finding the expression for M_J).

Out[13]=



d) Low mass collapse

Does the limit calculated in (c) mean that stars will form only with high mass?, what would be the physical process that leads to formation of less massive stars?. Why does a GMC with a total mass larger than this limit not collapse completely?

Observationally, we know that there are stars with mass much less that our limit calculated in (c)

The physical process that we think leads to the formation of low-mass stars in dense cores of GMCs is fragmentation. Any inhomogeneity in the density will cause that location to satisfy the Jean mass limit and collapse independently of the rest of the core, and hence fragmentation will occur forming multitude of smaller objects. But there are some unresolved issues, like: if this reasoning is correct the star formation efficiency will be quite large, but observations show that about 1% of the cloud actually form stars, and in this fragmentation scenario how do we form large stars? Although a GMC has a mass larger its Jean mass limit it would not collapse because there are other ways to support it against gravitation (like magnetic fields, rotation, turbulence, etc.) besides just thermal pressure.

e) Fragmentation

If the collapse of a cloud is isothermal the Jeans mass will decrease as the density increases, hence making it easier for different regions of the cloud to collapse independently and fragment. But it appears that there is no limit for how small a fragment will be, since as you get more and more dense you can fragment into smaller and smaller objects. The goal of this part is try to estimate the minimum mass for which fragmentation will be stopped.

- For each of these types of collapse, how does the Jeans mass scale with density?

$$M_J = \left(\frac{5\, k_B\, T}{G\, \mu\, m_H}\right)^{3/2} \left(\frac{3}{4\, \pi\, \rho}\right)^{1/2}$$

For an isothermal collapse: $M_J \propto \rho^{-1/2}$. For an adiabatic collapse: $T \propto \rho^{\gamma-1}$, then $M_J \propto \rho^{(3\gamma-4)/2}$

- Calculate what is the energy released by contraction and the timescale for free-fall.

The gravitational potential energy of a self-gravitating object is defined as the negative of the amount of energy that is required to disperse its mass to infinity, and can be written as:

$$\Omega = \int_0^M -\frac{G\, M_r}{r}\, dM_r$$

If we assume that the density is constant throughout the cloud, the previous integral is calculated as:

$$\Omega = \int_0^M -\frac{G}{r}\left(\frac{4}{3}\pi\, r^3\, \rho\right) 4\, \pi\, r^2\, \rho\, dr = \frac{-3}{5}\frac{G\, M^2}{R}$$

Free-fall timescale: The equation of motion for a test particle of mass *m* at a distance *r* of the center of the cloud of mass *M* and radius *R* is:

$$m\, \ddot{r} = -\frac{G\, M_r\, m}{r^2}$$

Assuming a constant density $\overline{\rho}$ throughout the cloud we can write$M_r = \frac{4\pi}{3} r^3 \overline{\rho}$, replacing this in the previous equation gives a harmonic oscillator:

$$\ddot{r} + \frac{4\, \pi\, G\, \overline{\rho}}{3}\, r = 0$$

with a characteristic frequency of oscillation $\omega = \sqrt{\frac{4\pi}{3} G\, \overline{\rho}}$. The typical time scale of the particle to fall into the gravitational potential of the cloud will be given by the inverse of the frequency of oscillation:

$$t_{ff} = \sqrt{\frac{3}{4\, \pi\, G\, \overline{\rho}}} \sim \frac{1}{\sqrt{G\, \overline{\rho}}}$$

- If the gravitational energy that is released during collapse can be radiated away efficiently (i.e. $L_{cloud} = L_{grav} \sim \Delta E_{grav}/t_{ff}$) you can keep the temperature nearly constant and have an isothermal collapse. But if you can't transport efficiently the energy out of the cloud (i.e. $L_{cloud} = e\, L_{rad}$, where $e < 1$ is an efficiency factor), the temperature will rise and the collapse will be adiabatic. The minimum mass of a fragment arises from the transition between isothermal and adiabatic collapse.
  Assuming that you are just in the transition point from isothermal to adiabatic, and further assume that the cloud is optically thick and in thermodynamic equilibrium, calculate the minimum Jean mass for a fragment. Evaluate for typical values: $T \sim 1000\, K$, $e \sim 0.1$.

Since we are at the transition point between adiabatic and isothermal collapse we will have that the cloud's luminosity will be equal to the "gravitational" luminosity and to the "radiative" luminosity:

$$L_{cloud} = \frac{\Delta E_{grav}}{t_{ff}} = e\, L_{rad}$$

where

$$L_{rad} = 4\,\pi\,R^2\,\sigma_B\,T^4$$

$$L_{grav} = \frac{3}{5}\,G^{3/2}\left(\frac{M}{R}\right)^{5/2}$$

Replacing the radius as a function of mass and density, and using the expression for the Jeans mass as a function of T and ρ we obtain:

$$M_{J,min} = 0.03\,\frac{T^{1/4}}{e^{1/2}\,\mu^{9/4}}\,M_\odot$$

Evaluating for typical values: $M_{J,\mathrm{min}} \sim 0.5\,M_\odot$

Worked Example: Bonnor-Ebert Mass

For a given surface pressure P_s and sound speed c_s there exists a maximum mass at which an isothermal cloud can be in hydrostatic equilibrium, called the Bonnor-Ebert mass. In a typical low-mass star-forming region, the surface pressure on a core might be P_s/k_B=3×10^5 K cm^{-3}. Compute this mass for a core with a temperature of 10 K, assuming the standard mean molecular weight $\mu = 3.9 \times 10^{-24}$g. What is the corresponding radius of the cloud?

Solution

From the lecture we know that

$$R = \frac{c_s^{\,2}}{\sqrt{2\,\pi\,G}}\,P_{\mathrm{ext}}^{\,-1/2} \tag{1}$$

$$c_s^{\,2} = \frac{p}{\rho} \tag{2}$$

from the ideal gas we know

$$p\,V = n_m\,R\,T = N\,k_B\,T \tag{3}$$

$$p = \frac{N}{V}\,k_B\,T = n\,k_B\,T \tag{4}$$

with n the number density of particles (n_m is the number of molecules).

$$p = \frac{N}{V}\,k_B\,T = n\,k_{B\,T} \tag{5}$$

$$\frac{p}{n} = k_B\,T \tag{6}$$

Since $\rho = n\,\mu$ we find

$$c_s^{\,2} = \frac{p}{\rho} = \frac{k_B\,T}{\mu} \tag{7}$$

inserting (7) into (1) and expanding with k_B/k_B gives

$$R_{\mathrm{BE}} = \frac{\left(\frac{k_B\,T}{\mu}\right)^2}{\sqrt{2\,\pi\,G}}\left(k_B\,\frac{P_{\mathrm{ext}}}{k_B}\right)^{-1/2} \tag{8}$$

In[19]:=
```
Radius[Poverk_, T_, μ_] := ((1.38066×10^-16 T)/μ)^2 / Sqrt[2 π 6.6726 × 10^-8] (1.38066 × 10^-16 Poverk)^(-1/2)
```

In[20]:=
```
Radius[3 × 10^5, 10, 3.9 × 10^-24] / (3.08 × 10^18)
```

Out[20]= 0.0275823

For the mass:

$$M_{BE} = \sqrt{\frac{2}{\pi}} \frac{c_s{}^4}{G^{3/2}} P_{ext}{}^{-1/2} \tag{9}$$

using the same substitutions we find:

$$M_{BE} = \sqrt{\frac{2}{\pi}} \frac{\left(\frac{k_B T}{\mu}\right)^2}{G^{3/2}} \left(k_B \frac{P_{ext}}{k_B}\right)^{-1/2} \tag{10}$$

```
Mass[Poverk_, T_, μ_] := Sqrt[2/π] ((1.38066 × 10^-16 T)/μ)^2 / (6.6726 × 10^-8)^(3/2) (1.38066 × 10^-16 Poverk)^(-1/2)
```

```
Mass[3 × 10^5, 10, 3.9 × 10^-24]
```

9.01442×10^{32}

and in units of solar masse:

```
Mass[3 × 10^5, 10, 3.9 × 10^-24] / (1.989 × 10^33)
```

0.453214

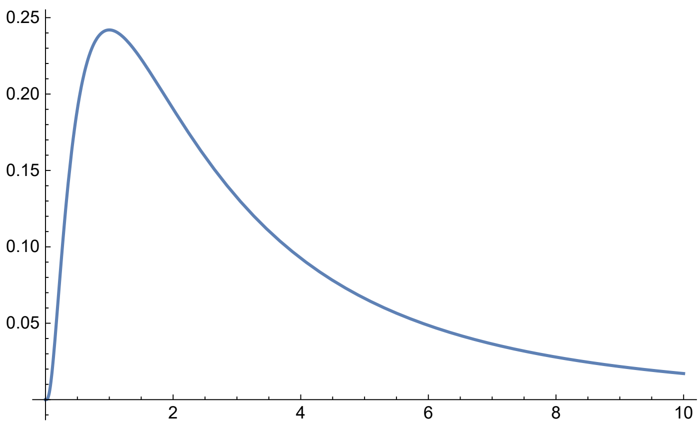
Mass tracer

Consider a molecular cloud in which the volume-averaged density is $n = 100\ \text{cm}^{-3}$. Assuming the cloud has a lognormal density distribution as given by :

$$f_{V,M} = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(x \pm |\mu_x|)^2}{2\sigma_x^2}\right) \tag{11}$$

In[21]:=
```
Plot[PDF[LogNormalDistribution[1, 1], x], {x, 0, 10}]
```

Out[21]=



were the fraction of volume (V) or mass (M) as a function of $x = \ln(n/\bar{n})$ is given by $f(x)\,dx$. The mean and dispersion of the distributions are related by $\mu_x = \sigma_x^2/2$, and the upper and lower signs correspond to volume- and mass-weighting, respectively.

With a mean $\mu_x = 3.0$, compute the fraction of the cloud mass that is denser than the critical density for each of the transitions from a).

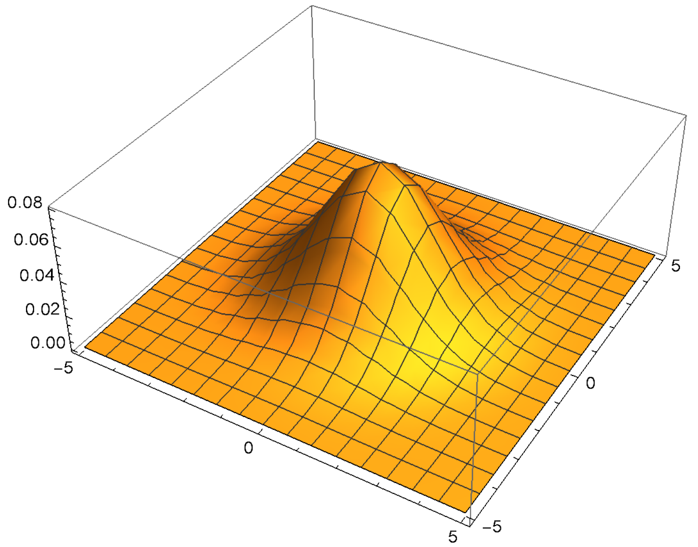
In the last problem set we calculated the critical densities of the following lines: CO J = 1→0, CO J = 4→3, CS J = 1→0, and HCN J = 1→ 0:

Line	n_{crit} [cm − 3]
CO J = 1 → 0	2.2×10^3
CO J = 4 → 3	3.9×10^4
CS J = 1 → 0	4.7×10^4
HCN J = 1 → 0	1.0×10^6

Which of theses transitions are good tracers of the bulk of the mass in a cloud? Which are good tracers of the denser, and thus presumably more actively star-forming, parts of the cloud?

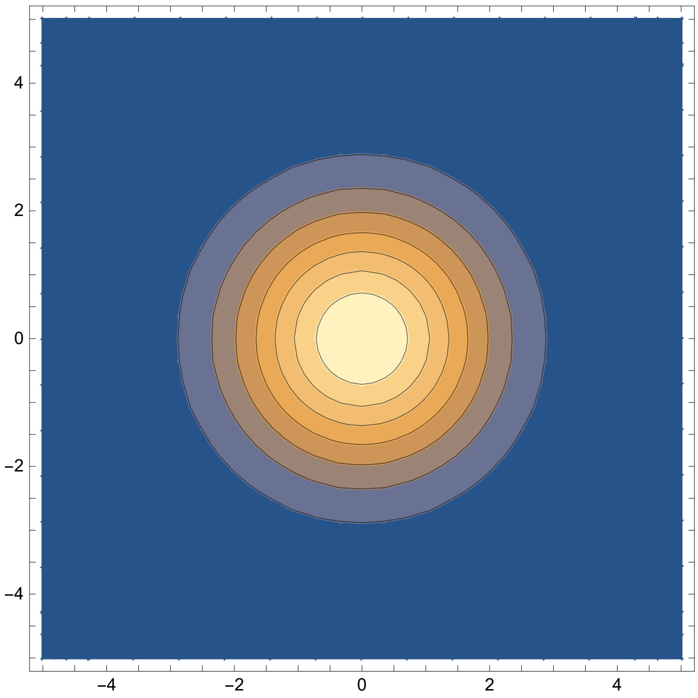
In[24]:=
```
Plot3D[PDF[MultinormalDistribution[{0, 0}, {{2, 0}, {0, 2}}], {x, y}],
 {x, -5, 5}, {y, -5, 5 }, PlotRange → All]
```

Out[24]=



In[25]:=
```
ContourPlot[PDF[MultinormalDistribution[{0, 0}, {{2, 0}, {0, 2}}], {x, y}],
 {x, -5, 5}, {y, -5, 5 }, PlotRange → All]
```

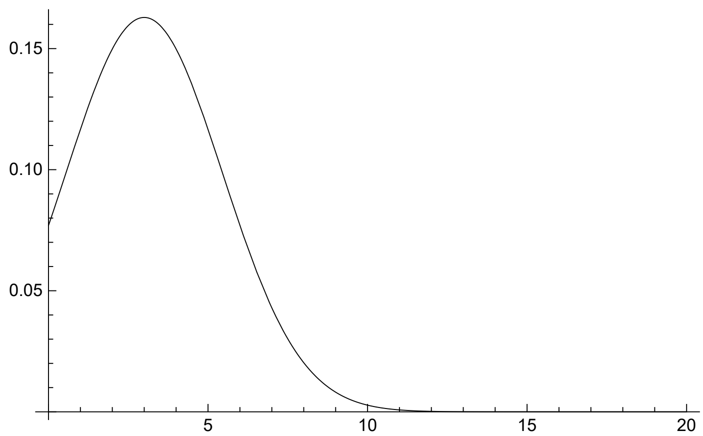
Out[25]=



Solution

Equation 11 gives the fraction of mass (volume) as a function of x. This fraction is normalized such that the integral over all x is equal to 1, so integrating between two values of x will give us the fraction of mass in the given density range. Integrating this equation from $x = \ln(n_{\rm crit}/\overline{n})$ to ∞, we obtain the following fractions of mass above the critical density for each molecular transition:

$$\texttt{Plot}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\,\texttt{Exp}\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\ /.\ \{\mu\to 3.,\ \sigma\to\sqrt{2\times 3.}\},\ \{x, 0, 20\}\right]$$



$$\int_{\rm xcrit}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\,\texttt{Exp}\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx$$

$$\texttt{ConditionalExpression}\left[\frac{1}{2}\sqrt{\frac{1}{\sigma^2}}\sqrt{\sigma^2}\ \texttt{Erfc}\left[\frac{(\texttt{xcrit}-\mu)\sqrt{\frac{1}{\sigma^2}}}{\sqrt{2}}\right],\ \texttt{Re}[\sigma^2] > 0\right]$$

```
fraction[ncrit_] := N[∫_{Log[ncrit/100]}^∞ 1/Sqrt[2 π 2 × 3] Exp[-(x - 3)^2/(2 × 2 × 3)] dx]
```

$$\text{fraction}[\text{ncrit\_}] := N\left[\int_{\text{Log}[\text{ncrit}/100]}^{\infty} \frac{1}{\sqrt{2\,\pi\,2\times3}}\,\text{Exp}\left[-\frac{(x-3)^2}{2\times2\times3}\right]dx\right]$$

fraction[2.2×10^3]

0.485176

fraction[3.9×10^4]

0.112962

fraction[4.7×10^4]

0.0990301

fraction[1.0×10^6]

0.00561658

The CO J = 1 → 0 transition is the best tracer of the bulk of the mass because its critical density is very close to the maximum in the log normal density distribution (10^3 cm^{-3}). The CS, HCN and uppermost CO transition are the best tracers of the densest regions of the cloud because their n_{crit} values are the largest, and thus they can probe denser regions of the cloud than the other transitions.